The Stieltjes matrix moment problem and associated positive symmetric operators

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The Stieltjes matrix moment problem. 1

The sequence of $m \times m$ matrices

$$s_0, s_1, \dots, s_l, \dots \subset \mathbb{C}^{m\times m} \qquad (1)$$

is called $\mathbb{R}_+$-positive if all block Hankel matrices

$$H_l^{(l)} = \begin{pmatrix} s_0 & \dots & s_l \\ \vdots & \ddots & \vdots \\ s_l & \dots & s_{2l} \end{pmatrix}, \quad H_2^{(l)} = \begin{pmatrix} s_1 & \dots & s_{l+1} \\ \vdots & \ddots & \vdots \\ s_{l+1} & \dots & s_{2l+1} \end{pmatrix}$$

are positive Hermitian.
Consider the matrix version of the Stieltjes moment problem:
Describe the set $\mathcal{M}_+$ of all nonnegative Hermitian $m \times m$ Borel measures σ on $\mathbb{R}_+$ such that

$$s_j = \int_{\mathbb{R}_+} t^j \sigma(dt), \qquad j \geq 0. \qquad (2)$$

Suppose the sequence (s_j) is $\mathbb{R}_+$-positive; then $\mathcal{M}_+ \neq \varnothing$.

Matrix polynomials

Suppose $\sigma \in \mathcal{M}_+$; then there exist two sequences of matrix polynomials

$$\left\{P_r^{(j)}(z)\right\}_{j=0}^{\infty}, \qquad r = 1, 2$$

such that:

1) Each polynomial $P_r^{(j)}$ is a matrix polynomial of degree j, and its leading coefficient is a positive $m \times m$ matrix.

2) For any matrix measure $\sigma \in \mathcal{M}_+$ the polynomials $P_r^{(j)}$ are orthonormal

$$\int_{\mathbb{R}_+} P_r^{(j)}(t) t^{r-1} \sigma(dt) P_r^{(k)^*}(t) = \delta_{jk} I_m, \quad \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Jacobi matrices

The matrix polynomials $\left(P_r^{(j)}\right)_{j=0}^{\infty}$ satisfy by the recurrence relation $(j \geq 1,\ r = 1, 2)$

$$tP_r^{(j)}(z) = B_r^{(j-1)^*} P_r^{(j-1)}(z) + A_r^{(j)} P_r^{(j)}(z) + B_r^{(j)} P_r^{(j+1)}(z) \quad (3)$$

and the initial condition

$$P_r^{(0)}(z) \equiv H_r^{(0)^{-1/2}}, \quad tP_r^{(0)}(z) = A_r^{(0)} P_r^{(0)}(z) + B_r^{(0)} P_r^{(1)}(z). \quad (4)$$

Here

$$A_r^{(j)} > O, \quad \det B_r^{(j)} \neq 0, \quad j \geq 0, \quad r = 1, 2.$$

From the coefficients of the recurrence relations we construct two infinite block Jacobi matrices

$$\mathbf{J}_r = \begin{pmatrix} A_r^{(0)} & B_r^{(0)} & O & O & \dots \\ B_r^{(0)^*} & A_r^{(1)} & B_r^{(1)} & O & \dots \\ O & B_r^{(1)^*} & A_r^{(2)} & B_r^{(2)} & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad r = 1, 2.$$

Associated Operators

Let $\ell_0^2(\mathbb{C}^m)$ be the subspace of $\ell^2(\mathbb{C}^m)$ which consists of finite vectors. We define two symmetric operators $\tilde{\mathbf{L}}_r : \ell_0^2(\mathbb{C}^m) \to \ell_0^2(\mathbb{C}^m),\ r = 1, 2$ as follows:

$$\tilde{\mathbf{L}}_r u = \mathbf{J}_r u, \quad \forall u \in \ell_0^2(\mathbb{C}^m).$$

These operators are nonclosed symmetric operators on $\ell_0^2(\mathbb{C}^m)$. Let $\mathbf{L}_r$ be their closures. The operators $\mathbf{L}_r,\ r = 1, 2$ will be said to be associated with the Stieltjes matrix moment problem.
The subspaces

$$\mathcal{D}_r(z) = \left\{\mathbf{u} \in \ell^2(\mathbb{C}^m) : \mathbf{L}_r^* \mathbf{u} = z\mathbf{u}\right\}, \quad r = 1, 2 \tag{5}$$

are called the deficiency subspaces of the operators $\mathbf{L}_r$ at z.

Theorem

1. Symmetric operators $\mathbf{L}_r,\ r=1,2$ *are non-negative.*
2. The dimension of the deficiency subspaces $\mathcal{D}_r(z)$ *are independent of the choice of the point* z *from* $\mathbb{C}\setminus\mathbb{R}_+$*:*

$$\dim\mathcal{D}_r(z_1)=\dim\mathcal{D}_r(z_2)=m_r\ \forall z_1,z_2\in\mathbb{C}\setminus\mathbb{R}_+,\ r=1,2.$$

3. There exist subspaces $\mathcal{L}_r(z)\subset\mathbb{C}^m,\ r=1,2$ *such that the mappings*

$$\phi\in\mathcal{L}_r(z)\longleftrightarrow\mathbf{u}=col\Big(P_r^{(0)}(z),P_r^{(1)}(z),P_r^{(2)}(z),\dots\Big)\phi\in\mathcal{D}_r(z),$$

give isomorphisms between the linear spaces $\mathcal{L}_r(z)$ *and* $\mathcal{D}_r(z)$*.*
4. The dimension of the subspaces $\mathcal{R}(z)=\mathcal{L}_1(z)\cap\mathcal{L}_2(z)$ *are independent of the choice of the point* z *from* $\mathbb{C}\setminus\mathbb{R}_+$

$$\dim\mathcal{R}(z_1)=\dim\mathcal{R}(z_2)=\delta\ \forall z_1,z_2\in\mathbb{C}\setminus\mathbb{R}_+.$$

Stieltjes matrix moment problem is naturally called:

a) completely indeterminate if $\delta = m$;

b) completely determinate if $\delta = 0$;

c) semi-determinate if $0 < \delta < m$.

A completely determinate or semi-determinate Stieltjes matrix moment problem will be called degenerate.

The Stieltjes criteria

Theorem

For the Stieltjes matrix moment problem to be nondegenerate, it is necessary and sufficient that the series

$$\sum_{j=0}^{\infty} P_1^{(j)^*}(x) P_1^{(j)}(x) < \infty, \quad \sum_{j=0}^{\infty} P_2^{(j)^*}(x) P_2^{(j)}(x) < \infty, \quad x \in \mathbb{R}_- \tag{6}$$

be convergent.

We have $\mathcal{L}_1(x) = \mathcal{L}_2(x) = \mathbb{C}^m$. This implies that the Stieltjes matrix moment problem is nondegenerate.
The classical Stieltjes criterion indeterminacy for the Stieltjes moment problem

$$\sum_{j=0}^{\infty} m_j < \infty, \; \sum_{j=0}^{\infty} \ell_j < \infty$$

is a special case of our theorem.

The Stieltjes matrix parameters.

Let the sequence (s_j) be $\mathbb{R}_+$-positive. We consider the block matrices:

$$H_r^{(l)} = (s_{j+k+r})_{j,k=0}^{l}, \ r = 1, 2,$$

$$v^{(0)} = \ (I) \ , \ v^{(j)} = \begin{pmatrix} v^{(j-1)} \\ O \end{pmatrix}, \ u^{(0)} = (s_0), \ u^{(j)} = \begin{pmatrix} u^{(j-1)} \\ s_j \end{pmatrix}.$$

The positive $m \times m$ matrices

$$\begin{aligned} m_0 &= v^{(0)^*} H_1^{(0)^{-1}} v^{(0)} > O, \\ \ell_0 &= u^{(0)^*} H_2^{(0)^{-1}} u^{(0)} > O, \\ m_j &= v^{(j)^*} H_1^{(j)^{-1}} v^{(j)} - v^{(j-1)^*} H_1^{(j-1)^{-1}} v^{(j-1)} > O, \\ \ell_j &= u^{(j)^*} H_2^{(j)^{-1}} u^{(j)} - u^{(j-1)^*} H_2^{(j-1)^{-1}} u^{(j-1)} > O. \end{aligned}$$

are called the Stieltjes matrix parameters.

Hamburger's theorem

Theorem

Let $(s_j)_{j=0}^{\infty}$ be an $\mathbb{R}_+$-positive sequence and $z \in \mathbb{C} \setminus \mathbb{R}_+$.
For the Hamburger moment problem

$$s_j = \int_{\mathbb{R}} t^j \sigma(dt), \quad j \geq 0$$

to be nondegenerate and for the Stieltjes moment problem

$$s_j = \int_{\mathbb{R}_+} t^j \tau(dt), \quad j \geq 0$$

to be degenerate, it is necessary and sufficient that the series

$$\sum_{j=0}^{\infty} P_1^{(j)^*}(z) P_1^{(j)}(z) \qquad \text{be convergent} \tag{7}$$

and the series

$$\sum_{j=0}^{\infty} P_2^{(j)^*}(z) P_2^{(j)}(z) \qquad \text{be divergent.} \tag{8}$$

Hamburger's theorem

We have

$$\mathcal{L}_1(z) = \mathbb{C}^m, \ \mathcal{L}_2(z) \subset \mathbb{C}^m, \ \dim \mathcal{L}_2(z) = m_2 < m;$$

It now follows that

$$\mathcal{R}(z) = \mathcal{L}_1(z) \cap \mathcal{L}_2(z) = \mathcal{L}_2(z)$$

and

$$\dim \mathcal{R}(z) = \dim \mathcal{L}_2(z) = m_2 < m.$$

This implies that the Stieltjes matrix moment problem is degenerate.
The classical Hamburger's theorem

$$\sum_{j=1}^{\infty} (\ell_0 + \ell_1 + \ldots + \ell_{j-1})^* \, m_j \, (\ell_0 + \ell_1 + \ldots + \ell_{j-1}) < \infty, \qquad \sum_{j=0}^{\infty} \ell_j > \infty.$$

is a special case of our theorem.