Controllability problems for the heat equation on a half-axis

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Consider the heat equation

$$w_t(x,t) = w_{xx}(x,t), \quad x \in (0,+\infty),\ w(0,t) = u(t), \tag{1}$$

controlled by the boundary condition

$$w(0,t) = u(t), \quad t \in (0,T), \tag{2}$$

under the initial condition

$$w(x,0) = w^0(x), \quad x \in (0,+\infty), \tag{3}$$

where $T > 0$ is given, $u \in L^\infty(0,T)$ is the control, the state $w(\cdot,t)$, $t \in (0,T)$, and the initial state w^0 belong to the space $H^0(0,+\infty)$ of the Sobolev type.

A state $w^0 \in H^0(0,+\infty)$ is called *approximately controllable* at a given time T if for any $w^T \in H^0(0,+\infty)$ and for any $\varepsilon > 0$ there exists a control $u_\varepsilon \in L^\infty(0,T)$ such that for the solution w_ε to system (1)–(3) with $u = u_\varepsilon$ we have $\left\|w^T - w_\varepsilon(\cdot,T)\right\| < \varepsilon$.

In the talk, it is shown that each state $w^0 \in H^0(0,+\infty)$ is approximately controllable at a given time T. The controls solving the approximate controllability problems are constructed.

For a state $w^0 \in H^0(0,+\infty)$, by $\mathcal{R}_T^1(w^0)$ denote a set of states $w^T \in H^0(0,+\infty)$ for which there exists a control $u \in L^\infty(0,T)$, $0 \le u(t) \le 1$, $t \in (0,T)$, such that for the solution w to system (1)–(3) we have $w(\cdot,T) = w^T$.

For states $w^0, w^T \in H^0(0,+\infty)$, we obtain necessary and sufficient conditions for $w^T \in \mathcal{R}_T^1(w^0)$. Under these conditions, using the Markov power moment problem, it is constructed a sequence $\{u_n\}_{n=1}^\infty$ of bang-bang controls ($u(t) \in \{0,1\}$, $t \in (0,T)$) such that for the solution w to system (1)–(3) with $u = u_n$ we have $\|w_n(\cdot,T) - w^T\| \to 0$ as $n \to \infty$.

These results are illustrated by examples.