Feedback linearizability in the class C^1

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Linear controllable systems

$$\dot{x} = Ax + bu(t), \quad x \in \mathbb{R}^n, \ u = u(t) \in \Omega \subset \mathbb{R}^1, \qquad x(0) = x^0, \quad x(T) = x^1; \tag{1}$$

$$\mathrm{rank}(b, Ab, \ldots, A^{n-1}b) = n.$$

- Controllability.
- Stabilizability.
- Optimal control.
- ...

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General systems:

$$\dot{x} = f(x, u). \tag{2}$$

Map to a linear one?

Linearizability of nonlinear systems

1. Nonlinear change of variables in a linear system.
   Example: in the system
   $$\dot{x}_1 = x_2, \quad \dot{x}_2 = u,$$
   let us set $z_1 = x_1$, $z_2 = x_2 + x_1^3$; then
   $$\dot{z}_1 = z_2 - z_1^3, \quad \dot{z}_2 = 3z_1^2(z_2 - z_1^3) + u.$$

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2. Some systems can be linearized by a change of variables *and controls*.

   Example: in the system

   $$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1^2 - x_1 x_2^3 + u,$$

   let us set $v = x_1^2 - x_1 x_2^3 + u$; then

   $$\dot{x}_1 = x_2, \quad \dot{x}_2 = v.$$

Local linearizability in a domain

A system

$$\dot{x} = f(x, u) \tag{3}$$

is called **locally linearizable in the domain** $Q \subset \mathbb{R}^n$ if these exists a change of variables

$$z = F(x), \quad \det F_x(x) \neq 0, \quad x \in Q, \tag{4}$$

such that

$$\dot{z} = Az + bu + c.$$

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- If the system is linearizable then $f(x, u) = a(x) + b(x)u$.
- If $a(x), b(x) \in C^\infty(Q)$ then $F(x) \in C^\infty(Q)$.

Local linearizability in a domain

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is called **feedback linearizable in the domain** $Q \subset \mathbb{R}^n$ if there exists a change of variables and of a control

$$\begin{aligned} z &= F(x), \quad \det F_x(x) \neq 0, \quad x \in Q, \\ v &= g(x, u), \quad g(x, \mathbb{R}) = \mathbb{R}, \quad g_u(x, u) \neq 0, \end{aligned} \tag{5}$$

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- If the system is feedback linearizable then $f(x, u) = a(x) + b(x)\phi(x, u)$.
- If $f(x, u) \in C^\infty$ then $F(x) \in C^\infty$, $g(x, u) \in C^\infty$.

Conditions of linearizability

- $$\dot{x} = a(x) + b(x)u.$$

  Conditions of linearizability?

Krener, A.: On the equivalence of control systems and the linearization of nonlinear systems. SIAM J. Control (1973)

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  Conditions of feedback linearizability?

  Korobov, V.I.: Controllability, stability of some nonlinear systems. Diff. Equat. (1973)

Triangular systems (V. I. Korobov, 1973)

Consider a system

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dots & \\ \dot{x}_{n-1} &= f_{n-1}(x_1, \dots, x_n) \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u). \end{cases} \tag{6}$$

Suppose:

- $f_i \in C^{n+1-i}(\mathbb{R}^{i+1})$, $i = 1, \dots, n$;
- $\left| \frac{\partial f_i}{\partial x_{i+1}} \right| \geq \alpha > 0$, $i = 1, \dots, n$.

Then the system (6) is feedback linearizable in $Q = \mathbb{R}^n$ (globally).

Example

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2, u) \end{cases} \tag{7}$$

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Set $z_1 \stackrel{def}{=} x_1$, then

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Set $z_1 \overset{def}{=} x_1$, then

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = f_1(x_1, x_2) \overset{def}{=} z_2, \\ \dot{z}_2 &= \frac{d}{dt}(f_1(x_1, x_2)) = \\ &= \frac{\partial f_1(x_1,x_2)}{\partial x_1} f_1(x_1, x_2) + \frac{\partial f_1(x_1,x_2)}{\partial x_2} f_2(x_1, x_2, u) \overset{def}{=} v. \end{aligned}$$

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$$\begin{aligned} z_1 &= F_1(x_1) = x_1 \\ z_2 &= F_2(x_1, x_2) = f_1(x_1, x_2) \\ v &= g(x, u) = \frac{\partial f_1}{\partial x_1} f_1 + \frac{\partial f_1}{\partial x_2} f_2 \end{aligned} \quad \overset{(7)}{\Rightarrow} \quad \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v. \end{cases}$$

Triangular systems

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Conditions of linearizablity (A. Krener, 1973)

The system

$$\dot{x} = a(x) + b(x)u, \quad a(x), b(x) \in C^\omega(U(0)) \qquad (8)$$

is linearizable in $Q = U(0)$ if and only if

- $\mathrm{rank}\{\mathrm{ad}_a^k b(0)\}_{k=0}^{n-1} = n$;
- all Lie brackets of the vector fields $\mathrm{ad}_a^k b(x)$, $k \geq 0$, at $x = 0$ equal zero.

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$\mathrm{ad}_a^0 b(x) = b(x)$,

$\mathrm{ad}_a^{k+1} b(x) = [a(x), \mathrm{ad}_a^k b(x)]$, $k \geq 0$,

$[a(x), b(x)] = b_x(x)a(x) - a_x(x)b(x)$.

Conditions of linearizability

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- $\mathrm{rank}\{\mathrm{ad}_a^k b(0)\}_{k=0}^{n-1} = n$;
- $[\mathrm{ad}_a^k b(x), \mathrm{ad}_a^j b(x)] = 0, \quad 0 \le k, j \le n,\ x \in U(0).$

Then $z = F(x)$ is of the form $F_i(x) = L_a^{i-1} F_1(x)$ where

$$(F_1(x))_x \mathrm{ad}_a^k b(x) = 0,\ k = 0, \ldots, n-2,$$

$$(F_1(x))_x \mathrm{ad}_a^{n-1} b(x) = const, \quad x \in Q.$$

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$L_a f(x) = f_x(x) a(x)$.

Conditions of feedback linearizability

The system

$$\dot{x} = a(x) + b(x)\phi(x,u), \quad a(x), b(x) \in C^{\infty}(U(0)), \quad (10)$$

$\phi_u(x,u) \neq 0$, is locally feedback linearizable in $Q = U(0)$ if and only if

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- $\mathrm{rank}\{\mathrm{ad}_a^k b(0)\}_{k=0}^{n-1} = n$;
- $[\mathrm{ad}_a^k b(x), \mathrm{ad}_a^j b(x)] = \sum_{i=0}^{j} \eta_{k,j}^i(x)\mathrm{ad}_a^i b(x)$, $0 \leq k < j \leq n-2$.

Local linearizability in the class C^1

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Local linearizability in a domain in the class C^1

The system

$$\dot{x} = f(x,u), \quad f(x,u) \in C^1(Q \times \mathbb{R}), \tag{11}$$

is called **local linearizable in the domain $Q \subset \mathbb{R}^n$** if there exists a change of variables and of a control

$$z = F(x) \in C^2(Q), \quad \det F_x(x) \neq 0, \quad x \in Q, \tag{12}$$

such that

$$\dot{z} = Az + bu + c.$$

- If the system is linearizable then $f(x,u) = a(x) + b(x)u$ where $a(x), b(x) \in C^1(Q)$.

Local linearizability in a domain in the class C^1

The system $\dot{x} = f(x,u), \quad f(x,u) \in C^1(Q \times \mathbb{R}),$ (12)

is called **locally feedback linearizable in the domain** $Q \subset \mathbb{R}^n$ if there exists a change of variables and of a control

$$z = F(x) \in C^2(Q), \quad \det F_x(x) \neq 0, \quad x \in Q,$$
$$v = g(x,u) \in C^1(Q \times \mathbb{R}), \quad g(x,\mathbb{R}) = \mathbb{R}, \quad g_u(x,u) \neq 0, \tag{13}$$

such that $\dot{z} = Az + bv.$

- If the system is feedback linearizable then $f(x,u) = a(x) + b(x)\phi(x,u)$ where $a(x), b(x) \in C^1(Q)$, $\phi(x,u) \in C^1(Q \times \mathbb{R})$.

Criterion of local linearizability in a domain

The system

$$\dot{x} = a(x) + b(x)u, \quad a(x), b(x) \in C^1(Q) \qquad (14)$$

is locally linearizable in the domain Q if and only if

1. $\mathrm{ad}_a^k b(x)$, $k = 0, \ldots, n$, exist and belong to the class $C^1(Q)$;
2. $\mathrm{rank}\{\mathrm{ad}_a^k b(x)\}_{k=0}^{n-1} = n$, $x \in Q$;
3. $[\mathrm{ad}_a^k b(x), \mathrm{ad}_a^i b(x)] = 0$, $x \in Q$, $0 \leq i, k \leq n$.

- This is a direct generalization of conditions for systems of the class C^∞.

Example. The system of the class C^1

$$\dot{x}_1 = x_2^2|x_2| + x_2, \quad \dot{x}_2 = \frac{x_3}{3x_2|x_2|+1}, \quad \dot{x}_3 = u,$$

in the domain $Q = \{x \in \mathbb{R}^3 : x_2 > -\frac{1}{\sqrt{3}}\}$. Then

$$b(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathrm{ad}_a b(x) = \begin{pmatrix} 0 \\ -\frac{1}{3x_2|x_2|+1} \\ 0 \end{pmatrix}, \ \mathrm{ad}_a^2 b(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

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Choose $F_1(x) = x_1 \in C^2(Q)$; then

$$z = F(x) = \begin{pmatrix} F_1(x) \\ L_a F_1(x) \\ L_a^2 F_1(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2^2|x_2| + x_2 \\ x_3 \end{pmatrix}$$

maps the system to the form $\dot{z}_1 = z_2, \ \dot{z}_2 = z_3, \ \dot{z}_3 = u.$

Local feedback linearizability in the class C^1

Conditions for the class C^∞ are not sufficient.

Example 1.

$$\dot{x}_1 = x_1|x_1| + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

in $Q = \{x : \|x\| < d\}$ satisfies the conditions for the class C^∞:

$$\mathrm{ad}_a^0 b(x) = b(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathrm{ad}_a b(x) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \ \mathrm{ad}_a^2 b(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\text{hence,} \quad [\mathrm{ad}_a^k b(x), \mathrm{ad}_a^i b(x)] = 0, \quad 0 \le k, i \le 2,$$

$$\mathrm{rank}\{b(0), \mathrm{ad}_a b(0), \mathrm{ad}_a^2 b(0)\} = 3.$$

However, this system **cannot be mapped to a linear one by use of a change of variables and a change of a control of the class C^1.**

Local feedback linearizability in the class C^1

Conditions for the class C^∞ are not necessary.

Example 2.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_1|x_1| + x_3|x_3| + u$$

in $Q = \{x : \|x\| < d\}$ is mapped to the linear systems by use of

$$z = F(x) = x, \quad v = g(x,u) = x_1|x_1| + x_3|x_3| + u \in C^1(Q \times \mathbb{R}).$$

However,

$$\mathrm{ad}_a b(x) = \begin{pmatrix} 0 \\ -1 \\ -2|x_3| \end{pmatrix},$$

hence, $\mathrm{ad}_a^2 b(x)$ **does not exist** if $x_1 \neq 0$, $x_3 = 0$.

Local feedback linearizability in the class C^1

Conditions for the class C^∞ are not necessary.

Example 2.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_1|x_1| + x_3|x_3| + u.$$

If we «correct» $\mathrm{ad}_a b(x)$:

$$\chi^1(x) = \mathrm{ad}_a b(x) + 2|x_3| b(x) = \begin{pmatrix} 0 \\ -1 \\ -2|x_3| \end{pmatrix} + 2|x_3| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

then one can find the Lie bracket:

$$[a(x), \chi^1(x)] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Criterion of local feedback linearizability in a domain

The system $\dot{x} = a(x) + b(x)\phi(x, u)$, $a, b, \phi \in C^1$, is locally feedback linearizable in the domain Q if and only if

1. $\chi^0(x) = b(x)$, $\chi^k(x) = [a(x), \chi^{k-1}(x)] + \sum_{j=0}^{k-1} \mu_{kj}(x)\chi^j(x)$ exist and belong to $C^1(Q)$ where $\mu_{kj}(x) \in C(Q)$;
2. $\mathrm{rank}\{\chi^0(x), \ldots, \chi^{n-1}(x)\} = n$;
3. $[\chi^k(x), \chi^j(x)] = \sum_{i=0}^{k} \eta^i_{kj}(x)\chi^i(x)$, $0 \leq j < k \leq n-2$;
4. $\varphi_x(x)\chi^k(x) = 0$, $k=0, \ldots, n-2$, $\varphi_x(x)\chi^{n-1}(x) \neq 0$, where $L_a^{i-1}\varphi(x) \in C^2(Q)$, $i = 1, \ldots, n$.

Algorithm of finding $\chi^k(x)$

- Step 0: $\chi^0(x) = b(x)$.

Algorithm of finding $\chi^k(x)$

- Step 0: $\chi^0(x) = b(x)$.
- Step k: Suppose $\chi^0(x), \ldots, \chi^{k-1}(x) \in C^1(Q)$.
  Find $[a(x), \chi^{k-1}(x)]$ and find $\mu_{kj}(x) \in C(Q)$ so that
  $$\chi^k(x) = [a(x), \chi^{k-1}(x)] + \sum_{j=0}^{k-1} \mu_{kj}(x)\chi^j(x) \in C^1(Q).$$
  If it is possible, go to the next step.

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  If it is possible, go to the next step.
- Step n: Check whether there exists a solution
  $\varphi_x(x)\chi^k(x) = 0, \; k=0, \dots, n-2, \quad \varphi_x(x)\chi^{n-1}(x) \neq 0,$
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  Find $[a(x), \chi^{k-1}(x)]$ and find $\mu_{kj}(x) \in C(Q)$ so that
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  such that $\quad L_a^{i-1}\varphi(x) \in C^2(Q), \ i = 1, \ldots, n.$
- Change: $z_i = L_a^{i-1}\varphi(x), \ v = L_a^n\varphi(x) + L_b L_a^{n-1}\varphi(x)u.$

The algorithm is well defined:

- Uniqueness of $\mu_{kj}(x)$:

$$\mu_{kj}(x) - \widetilde{\mu}_{kj}(x) \in C^1(Q),\ j = 0, \ldots, k-1.$$

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- A method of finding $\mu_k(x) = \{\mu_{kj}(x)\}_{j=0}^{k-1}$:

$$\mu_k(x) = -(\Gamma_k(x))^{-1}\psi_k(x)$$

where

$$\Gamma_k(x) = \{\langle \chi^i(x), \chi^j(x)\rangle\}_{i,j=0}^{k-1},$$

$$\psi_k(x) = \{\langle [a(x), \chi^k(x)], \chi^j(x)\rangle\}_{j=0}^{k-1}.$$

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where

$$\Gamma_k(x) = \{\langle \chi^i(x), \chi^j(x)\rangle\}_{i,j=0}^{k-1},$$

$$\psi_k(x) = \{\langle [a(x), \chi^k(x)], \chi^j(x)\rangle\}_{j=0}^{k-1}.$$

- If the k-th step failed the system is not feedback linearizable.

Example

$$\dot{x}_1 = x_2 + x_2|x_2|, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_3|x_3| + u$$

in the domain $Q = \{x \in \mathbb{R}^3 : \|x\| < d\}$.

- Step 0: $\chi^0(x) = b(x) = (0, 0, 1)$.
- Step 1: Since $[a(x), \chi^0(x)] = \begin{pmatrix} 0 \\ -1 \\ -2|x_3| \end{pmatrix}$, we set

$$\chi^1(x) = [a(x), \chi^0(x)] + \mu_{00}(x)\chi^0(x) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

where $\mu_{00}(x) = 2|x_3| \in C(Q)$.

Example (*continued*)

Hence,

$$\chi^0(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \chi^1(x) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

- Step 2: Since $[a(x), \chi^1(x)] = \begin{pmatrix} 1 + 2|x_2| \\ 0 \\ 0 \end{pmatrix}$, for any $\mu_{10}(x)$ and $\mu_{11}(x)$ we get

  $$\chi^2(x)=[a(x), \chi^1(x)]+\mu_{10}(x)\chi^0(x)+\mu_{11}(x)\chi^1(x)\notin C^1(Q).$$

  Hence, step 2 failed. Hence, the system is not feedback linearizable.