Controllability of second-order partial differential equations in time

Alexander Makarov, *Kharkiv, Ukraine*
Tatyana Solodova, *Kharkiv, Ukraine*

Consider the following Cauchy problem

$$\begin{gathered}\frac{\partial^2 w(x,t)}{\partial t^2} + P\left(\frac{\partial}{\partial x}\right)\frac{\partial w(x,t)}{\partial t} + Q\left(\frac{\partial}{\partial x}\right) w(x,t) = u(x)v(t), \\ w(x,0)=\varphi(x), \quad w'_t(x,0)=\psi(x),\end{gathered} \tag{1}$$

where $P\left(\frac{\partial}{\partial x}\right)$ and $Q\left(\frac{\partial}{\partial x}\right)$ — differential operators with constant coefficients, $v(t)$ — piecewise continuous function on a segment $[0;T]$ and functions $u(x)$, $\varphi(x)$, $\psi(x)$ belong to the Schwartz space S. We seek for a control $u(x)v(t)$ such that for all $t \in [0;T]$ the solution $w(x,t)$ belongs to S and condition $w(x,T)=0$ is fulfilled.

Suppose $\lambda_1(s)$ and $\lambda_2(s)$ are roots of the characteristic equation $\lambda^2 + P(is)\lambda + Q(is) = 0$.

Then the Cauchy function for the Fourier transformed equation $\widetilde{w}''_{tt}(s,t) + P(is)\widetilde{w}'_t(s,t) + Q(is)\widetilde{w}(s,t) = \widetilde{u}(s)v(t)$ is as follows:

$$K(s,t,\tau) = \frac{\left(e^{\lambda_1(s)(t-\tau)} - e^{\lambda_2(s)(t-\tau)}\right)}{\lambda_1(s)-\lambda_2(s)}.$$

The controllability condition of equation (1) will look like this $R(s,T)=\int_0^T K(s,T,\tau)v(\tau)d\tau \neq 0,\ \forall s \in \mathbb{R}^n$. The following results are valid.

Theorem 1. *If the roots of the characteristic equation $\lambda_j(s)$ are real and at least one of them is bounded from above, then a control with $v(t)=1$ and $u(x) \in S$ such that equation (1) is controllable in the space S exists.*

Theorem 2. *If the roots of the characteristic equation $\lambda_j(s)$ are imaginary, then a control of the form $e^{\gamma(T-t)}u(x)$, where $u(x) \in S$, such that equation (1) is controllable in the space S with some $\gamma > 0$ exists.*

Theorem 3. *If conditions $Q(is) = 0$ and $ReP(is) \le c\ \forall s \in \mathbb{R}^n$ are satisfied, then equation (1) is controllable in the space S with $v(t) = 1$.*

`Example`. Equation $\frac{\partial^2 w(x,t)}{\partial t^2} - a^2 \triangle w(x,t) - kw(x,t) = u(x)v(t)$ is controllable in the space S with control of the form $e^{\gamma(T-t)}u(x)$ with some $\gamma > 0$.