On exact controllability and complete stabilizability for linear systems in Hilbert space

Rabah Rabah
IMT Atlantique, Mines-Nantes, France

October 19, 2018

Abstract

We consider linear systems in the general form

$$\dot{x} = \mathscr{A}x + \mathscr{B}u,$$

where the state $x(t)$ and the control $u(t)$ take values in Hilbert spaces X and U. $\mathscr{A}$ is a linear operator, infinitesimal generator of a C_0-semigroup $S(t)$, $\mathscr{B}$ is linear bounded operator.

By exact (null) controllability we mean controllability from any state to any state (or zero state). By complete stabilizability we mean exponential stabilizability with arbitrary decay rate or, sometimes pole assignment, by linear state feedback.

It is well known that in an finite dimensional setting exact controllability (said complete controllability) is a necessary and sufficient condition for complete stabilizability or more precisely arbitrary pole assignment. The situation is more complicated in infinite dimensional spaces.

We recall some classical results concerning the relation between exact controllability and complete stabilizability.

The first important result in this context was given by Slemrod: if $S(t)$ is a group, exact controllability implies complete stabilizability. The converse, for a group, was proved by Zabczyk. The result was generalized and precized by several authors for the case of a bounded operator $\mathscr{A}$, for the case of a semigroup $S(t)$ (not a group) and for some classes of systems, governed by partial differential equations or functional-differential equations.

We discuss more precisely the relations between exact null controllability and complete stabilizability. In general, for linear systems in Hilbert spaces, exact null controllability implies complete stabilizability, but the converse is not true. We give more recent results on functional-differential systems of neutral type.

1 Introduction

The well known relation between complete controllability and pole assignment may be formulated in the following form.

Theorem 1.1. *[7] The system*

$$\dot{x} = Ax + Bu \tag{1}$$

is completely controllable, i.e. $\operatorname{rank}[B\ AB \cdots A^{n-1}B] = n$*, if and only if the system is completely stabilizable, i.e. for all* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$*, there exists* F *such that the spectrum of the closed loop system verifies:*

$$\sigma(A + BF) = \{\lambda_1, \ldots, \lambda_n\},$$

i.e. the problem of pole assignment by linear feedback is solvable.

Another formulation of the same statment is that all the poles are controllable (Hautus criteria):

$$\forall \lambda \in \sigma(A), \quad \operatorname{rank}[\lambda I - A \quad B] = n$$

Our purpose is to consider the same problem in infinite dimensional Hilbert or Banach spaces. As the spectrum is more complicated and the behavior of the system is not completely determinated by the spectrum, solutions of such problem, when there exists some solutions, is not so simple. We will give a short review of different results and consider the special case of neutral type systems.

2 Exact controllability gives complete stabilizability

2.1 The case of bounded control operator $\mathscr{B}$

Consider the system

$$\dot{x} = \mathscr{A} x + \mathscr{B} u \tag{2}$$

in Hilbert spaces X and U, where the operator $\mathscr{A}$ is the infinitesimal generator of a group $\mathrm{e}^{\mathscr{A} t}$, $\mathscr{B}$ is a bounded operator from U to X.

Theorem 2.1. *The condition*

$$\exists \delta > 0,\ \forall x \in X : \quad \int_0^T \|\mathscr{B}^* \mathrm{e}^{-\mathscr{A}^* \tau} x\|^2 \, \mathrm{d}\tau \geq \delta \|x\|^2 \tag{EC}$$

implies complete stabilizability [6]:

$$\forall \omega > 0, \exists \mathscr{F} \in \mathscr{L}(X, U), \quad \|\mathrm{e}^{(\mathscr{A} + \mathscr{B}\mathscr{F})t}\| \leq M_\omega \mathrm{e}^{-\omega t}, \quad M_\omega \geq 1, \tag{CS}$$

with a linear bounded feedback $\mathscr{F}$ given by $\mathscr{F} = -\mathscr{B}^* K_\omega^{-1}$, where

$$K_\omega = \int_0^T \mathrm{e}^{-2\omega t} \mathrm{e}^{-\mathscr{A}\tau} \mathscr{B}\mathscr{B}^* \mathrm{e}^{-\mathscr{A}^*\tau} \, \mathrm{d}\tau.$$

This operator K_ω is, in fact, the gramian of the system

$$\dot{x} = \mathscr{A}x + \omega x + \mathscr{B}u$$

which the group is $\mathrm{e}^{\omega t}\mathrm{e}^{\mathscr{A}t}$.

In fact, the condition (EC) is the condition of exact controllability in the case of a group, and corresponds to the Kalman integral criterion:

$$K_0 = \int_0^T \mathrm{e}^{-\mathscr{A}\tau} \mathscr{B}\mathscr{B}^* \mathrm{e}^{-\mathscr{A}^*\tau} \, \mathrm{d}\tau$$

is bounded invertible. The proof follows Lyapunov 2d method. We can formulate this result as follows.

Theorem 2.2. *The system (2), where $\mathscr{A}$ is the infinitesimal generator of a group, then exact controllability (EC) implies complete stabilizability (CS).*

2.2 The case of unbouned $\mathscr{B}$

Following Slemrod and using some results by Lions, Komornik [2] proved an analogous result for the case when $\mathscr{B}$ is unbounded. This was applied to system governed by PDE with boundary control.

Shortly, with some conditions on the operator $\mathscr{B}$, complete stabilizability is obtained from exact controllability and the feedback is

$$\mathscr{F} = -J\mathscr{B}^* K_\omega^{-1}, \qquad K_\omega = \int_0^T \mathrm{e}^{-2\omega t} \mathrm{e}^{-\mathscr{A}\tau} \mathscr{B} J \mathscr{B}^* \mathrm{e}^{-\mathscr{A}^*\tau} \, \mathrm{d}\tau,$$

where J is the canonical Riesz isomorphism.

2.3 Application

This allow to obtain the complete stabilization of systems like

$$\ddot{w}(t,\xi) - \nabla w(t,\xi) = 0, \qquad (t,\xi) \in \mathbb{R}^+ \times \Omega$$

$$w(0,\xi) = f(\xi),\ \dot{w}(0,\xi) = g(\xi), \qquad \xi \in \Omega$$

$$w(t,\xi) = u(t,\xi) \qquad \xi \in \partial\Omega.$$

The boundary control is given by the formula

$$u(t,\xi) = \mathscr{F}w(t,\xi) = -J\mathscr{B}^* K_\omega^{-1}(w(t,\xi),\dot{w}(t,\xi))$$

defined in the space $H_{-1}(\Omega) \times L_2(\Omega)$:

$$u(t,\xi) = \partial_\nu(Pw(t,\xi) + Q\dot{w}(t,\xi)),$$

where P and Q are linear and bounded.

2.4 The case of a semigroup

In the general case of a semigroup, we have the following general implication:

Theorem 2.3. *For the system (2), with bounded operator $\mathscr{B}$, exact controllability to zero implies complete stabilizability by bounded feedback $\mathscr{F}$.*

This is a consequence of a the following result contained implicitly in a paper by Datko (1972) (see, for example [9]).

Theorem 2.4. *Null exactly controllable systems are exponentially stabilizable.*

This and the following remark give Theorem 2.3.

Remark 2.5. *Controllability of the system (2) is equivallent to controllability of the shifted system:*

$$\dot{x} = \mathscr{A}x + \omega x + \mathscr{B}u.$$

3 Complete stabilizability implies exact (null) controllability

3.1 In Banach space, bounded case

Consider the system

$$\dot{x} = Ax + Bu, \tag{3}$$

where operators A and B are bounded, in Banach spaces U, X. This means that it is not the case for PDE or functional-differential equations (delay equations). Anyway, it is of historical and theoretical interest.

An important result was obtained in [5].

Theorem 3.1 (Sklyar, 1982)**.** *The system (3) with bounded operators is completely stabilizable if and only if the block operator $[B \;\; AB \;\; \cdots \;\; A^kB]$ is right invertible for some k:*

$$\exists k \in \mathbb{N}, \quad \exists P_i \in \mathscr{L}(U,X) \qquad BP_0 + ABP_1 + \cdots + A^kBP_k = I. \tag{4}$$

If the spaces X, U are Banach spaces, not isomorphic to Hilbert spaces, the condition (4) is more that exact controllability, it is equivalent to the following two conditions:

1. $\operatorname{Im}[B \ AB \ \cdots \ A^k B] = X$, i.e. exact controllability according to the result by Korobov and Rabah [3],
2. $\operatorname{Ker}[B \ AB \ \cdots \ A^k B]$ is complemented.

In Banach spaces there is exactly controllable system, generated by a group, which are not completely stabilizable.

The situation is different in Hilbert spaces: every closed subspace can be complemented. And we have the following range inclusion result.

Theorem 3.2 (Douglas 1966)**.** *Let C and D be linear bounded operator in Hilbert spaces. The following 2 conditions are equivalent:*

1. $\operatorname{Im} C \subset \operatorname{Im} D$,
2. *There is a linear operator bounded E such that $C = DE$.*

In general, it is not true in Banach spaces.

3.2 The bounded case: pole assignment in Hilbert spaces

Important result which may be interesting, at least in operator theory.

In Hilbert spaces, exact controllability is equivalent to complete stabilizability, but more precisely we have a pole assignment result.

Theorem 3.3 (Eckstein, 1981 [1])**.** *The system (3) is exactly controllable if and only if for every nonempty compact set $\Lambda \subset \mathbb{C}$, there is a bounded linear feedback F, such that*

$$\sigma(A + BF) = \Lambda.$$

The proof uses explicitly the Hilbert space structure, in particular the fact that the subspace $\operatorname{Ker}[B \ AB \ \cdots \ A^k B]$ is complemented.

3.3 Unbounded case in Hilbert spaces, I

We return to the system (2):

$$\dot{x} = \mathscr{A}x + \mathscr{B}u, \tag{2}$$

with unbounded operator $\mathscr{A}$ and bounded operator $\mathscr{B}$.

Theorem 3.4 (Megan, Zabczyk, 1976 [8]). *If* $e^{\mathscr{A}t}$ *is a group and the system (2) is completely stabilizable, then it is exactly controllable.*

This result was extended to the case of surjective operators $e^{\mathscr{A}t}$, $t \geq 0$ by Rabah & Karrakchou (1997) and Zeng & Xie & Guo (2013).

Then we can summarize as:

Theorem 3.5 (All). *The system (2) exactly controllable with a bounded operator* $\mathscr{B}$ *if and only if*

1. $e^{\mathscr{A}t}$ *are surjective for* $t \geq 0$*,*
2. *The system is completely stabilizable.*

3.4 Unbounded case in Hilbert spaces, II

We consider the same system (2) with possible unbounded $\mathscr{B}$ and $\mathscr{F}$, they are supposed to be admissible in a certain sense:

$\mathscr{B}$ bounded from U to the space X_{-1} which is the completion of X with the norm $\|(\lambda I - A)^{-1}x\|$.

Theorem 3.6 ([4]). *If system (2) with admissible operator* $\mathscr{B}$ *is exactly null controllable in* X_{-1}*, then it is completely stabilizable by an admissible feedback and then by a bounded feedback* $\mathscr{F}$*.*

The converse is not true. We have different situations.

3.5 Example 1

In the space $L_2(0, +\infty)$, consider the semigroup

$$S(t)f(\xi) = e^{-\frac{t^2}{2} - \xi t} f(\xi + t), \quad t \geq 0, \quad \xi \geq 0.$$

It is not difficult to see that for this semigroup

$$\forall \omega > 0, \quad \exists M_\omega \geq 1, \quad \|S(t)\| \leq M_\omega e^{-\omega t}.$$

We have $\sigma(S(t)) = \{0\}$ and then $\sigma(\mathscr{A}) = \emptyset$.

Trivial that, for some $f \neq 0$ we have $S(t)f \neq 0$ for any $t \geq 0$.

The system is completely stabilizable ($\mathscr{B} = 0$) but not controllable.

3.6 Example 2

In the space $L_2(0,1)$, consider the semigroup

$$S(t)f(\xi) = \begin{cases} f(\xi+t) & 0 \le t+\xi \le 1, \\ 0 & t+\xi > 1. \end{cases}$$

It is not difficult to see that for this semigroup

$$\forall \omega > 0, \quad \exists M_\omega \ge 1, \quad \|S(t)\| \le M_\omega \mathrm{e}^{-\omega t}.$$

We have $\sigma(S(t)) = \{0\}$ and then $\sigma(\mathscr{A}) = \emptyset$.

For any initial function $f \in L_2(0,1)$, we have $S(t)f(x) = 0$ for $t > 2$. This means that $S(t) = 0$, for all $t > 2$.

Then, for any control operator $\mathscr{B}$, the system is exactly null controllable at time $T > 2$ with the trivial control $u = 0$.

4 The case of neutral type systems

4.1 Neutral type systems

We consider some large class of neutral type systems

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\mathrm{d}\theta + Bu, \qquad (5)$$

where $z(t)$ takes values in $\mathbb{R}^n$ and which can be written as

$$\dot{x} = \mathscr{A}x + \mathscr{B}u,$$

with finite dimensional and bounded $\mathscr{B}$.

The charasteristic matrix is noted by

$$\Delta_{\mathscr{A}}(\lambda) = \lambda I - A_{-1}\mathrm{e}^{-\lambda} - \lambda \int_{-1}^{0} A_2(\theta)\mathrm{e}^{s\theta}\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta))\mathrm{e}^{s\theta}\mathrm{d}\theta.$$

The behavior of at infinity are described by the spectrum:

$$\sigma(\mathscr{A}) = \{\lambda : \det \Delta_{\mathscr{A}}(\lambda) = 0\}.$$

4.2 Complete stabilizability

It is characterized by two conditions: one on the spectrum $\sigma(\mathscr{A})$, the second on the neutral term (here A_{-1}).

Theorem 4.1. *The system (5) is completely stabilizable if and only if*

1. *For all $\lambda \in \mathbb{C}$,* $\quad \text{rank}\begin{bmatrix}\Delta_{\mathscr{A}}(\lambda) & B\end{bmatrix} = n$,
2. *For all $\mu \in \mathbb{C}$, $\mu \neq 0$,* $\quad \text{rank}\begin{bmatrix}\mu I - A_{-1} & B\end{bmatrix} = n$.

It is important to check the meaning of the second condition:

The spectrum of the operator $\mathscr{A}$ is close to some vertical axes defined by the spectrum of the neutral term, the matrix A_{-1}. These axes must be moved by the feedback, then the corresponding values μ have to be controllable.

A large part of the spectrum of $\mathscr{A}$ may be obtained from the non zero spectrum of the matrix A_{-1}:

$$\lambda \quad = \quad \log|\mu| + \mathrm{i}(\arg\mu + 2k\pi) + o(1/|k|), \quad k \in \mathbb{Z}$$

For k large, the circles are centered at $\log\mu$ and are decreasing.

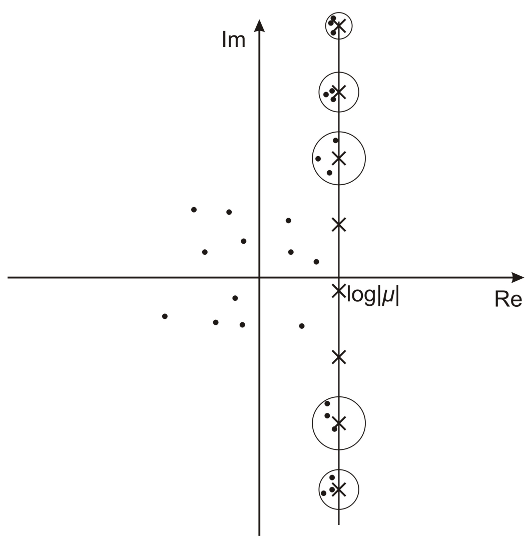

Figure 1: Spectrum of $\mathcal{A}$, $\mu \in \sigma(A_{-1})$

4.3 Complete stabilizability and exact controllability

Theorem 4.2. *If $\mathscr{A}$ is the infinitesimal generator of a group or, equivalently, if the matrix A_{-1} is not singular, $0 \notin \sigma(A_{-1})$, then exact controllability is equivalent to complete stabilizability.*

In fact, we have some result about pole assignment. Exact controllable implies that all the vertical axes can be moved, and in the circles, all the pole are controllable (may be arbitrary moved in the circles).

Theorem 4.3. *Exact controllability to zero implies complete stabilizability. The converse is proved in some particular cases (not distributed delays).*

In the case of our general system (5), the question is still open.

5 Conclusion

We vive a short review of the relations between exact controllability and pole assignment (complete stabilizability). There is lot of papers concerning different classes of systems, in particular, governed by PDE. This paper gives our point to view in the questions, based on results close to ours. Several questions are still open concerning the general abstract case or some general particular classes.

References

[1] Gheorghe Eckstein. Exact controllability and spectrum assignment. In *Topics in modern operator theory (Timişoara/Herculane, 1980)*, volume 2 of *Operator Theory: Adv. Appl.*, pages 81–94. Birkhäuser, Basel-Boston, Mass., 1981.

[2] Vilmos Komornik. Rapid boundary stabilization of linear distributed systems. *SIAM J. Control Optim.*, 35(5):1591–1613, 1997.

[3] V. I. Korobov and R. Rabah. Exact controllability in Banach space. *Differential Equations*, 15:1531–1537, 1980. Translated from russian : Differentsialnye Uravnenia, 15:2142-2150 (1980).

[4] Rabah Rabah, Grigory Sklyar, and Pavel Barkhayev. Exact null controllability, complete stabilizability and continuous final observability of neutral type systems. *Int. J. Appl. Math. Comput. Sci.*, 27(3):489–499, 2017.

[5] G.M. Sklyar. The problem of the perturbation of an element of a Banach algebra by a right ideal and its application to the question of the stabilization of linear systems in Banach spaces. *Vestn. Khar'kov. Univ.*, 230:32–35, 1982.

[6] Marshall Slemrod. A note on complete controllability and stabilizability for linear control systems in Hilbert space. *SIAM J. Control*, 12:500–508, 1974.

[7] W. Murray Wonham. *Linear multivariable control: a geometric approach.* Springer, New York, 3rd edition, 1985.

[8] J. Zabczyk. Remarks on the algebraic Riccati equation in Hilbert space. *Appl. Math. Optim.*, 2(3):251–258, 1975/76.

[9] Jerzy Zabczyk. *Mathematical control theory: an introduction.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.