The integrable nonlocal nonlinear Schrödinger equation: Riemann-Hilbert approach and long-time asymptotics

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Nonlocal nonlinear Schrödinger equation (NNLS)

We consider the "step-like" Cauchy problem

$$\begin{cases} iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0, & -\infty < x < \infty, t > 0 \\ q(x,0) = q_0(x), & -\infty < x < \infty, \end{cases}$$

where $q_0(x) \to 0$ as $x \to -\infty$ and $q_0(x) \to A$ as $x \to +\infty$ with some $A > 0$, with boundary conditions (for all $t \geq 0$)

$$q(x,t) = \begin{cases} o(1), & x \to -\infty \\ A + o(1), & x \to \infty \end{cases}$$

Recall the classical (local) NLS:

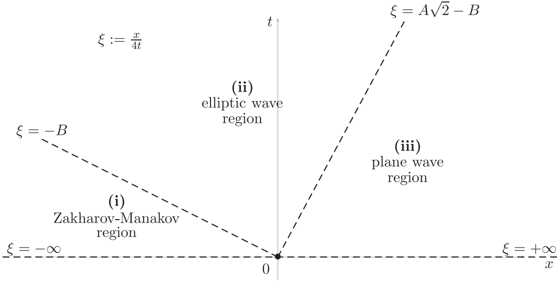
$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(x,t) = 0.$$

One can consider more general boundary conditions (for both NLS and NNLS)

$$q(x,t) = \begin{cases} o(1), & x \to -\infty \\ Ae^{2iBx+4i\omega t} + o(1), & x \to \infty; \quad A > 0,\ B, \omega \in \mathbb{R} \end{cases}$$

but notice that the relationships amongst A, B, and ω are different for NLS and NNLS: for NLS, $\omega = A^2/2 - B^2$; for NNLS, $\omega = -B^2$.

In the case $q_0(x) \to A\mathrm{e}^{-2\mathrm{i}Bx}$ as $x \to +\infty$, $q_0(x) \to 0$ $x \to -\infty$:



Three sectors in the (x,t) half-plane, where $q(x,t)$ behaves differently for large t, depending on the magnitude of $\xi = x/4t$.

- i $\xi < -B$: slowly decaying ($t^{-1/2}$) self-similar wave, as in the case of zero background
$$q(x,t) = \frac{1}{\sqrt{t}}\rho(-\xi)\mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + O(t^{-1})$$
- ii $-B < \xi < -B + A\sqrt{2}$: oscillations governed by modulated elliptic wave
- iii $\xi > -B + A\sqrt{2}$: plane wave
$$q(x,t) = A\mathrm{e}^{2\mathrm{i}(\omega t - Bx - \phi(\xi))} + O(t^{-1/2})$$

Inverse scattering transform method, I

Main goal: the large time analysis of the Cauchy problem for NNLS.

NNLS is an integrable nonlinear equation: it is the compatibility condition for two linear (matrix) equations (Lax pair):

$$\begin{cases} \Phi_x + ik\sigma_3\Phi = U(x,t)\Phi \\ \Phi_t + 2ik^2\sigma_3\Phi = V(x,t,k)\Phi \end{cases}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\Phi(x,t,k)$ is 2×2 matrix, $k \in \mathbb{C}$ is the spectral parameter,

$$U(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(-x,t) & 0 \end{pmatrix}, \; V(x,t,k) = 2kU(x,t) + \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & -\tilde{A} \end{pmatrix},$$

$\tilde{A} = iq(x,t)\bar{q}(-x,t)$, $\tilde{B} = iq_x(x,t)$, $\tilde{C} = i(\bar{q}(-x,t))_x$.

General scheme of the Inverse Scattering Transform method:

- $q(x,0) \to s(0,k)$: direct scattering problem;
- $s(0,k) \to s(t,k)$: (linear) evolution of scattering data;
- $s(t,k) \to q(x,t)$: inverse scattering problem: can be treated as a Riemann–Hilbert problem.

Inverse scattering transform method, II

$$\begin{cases} \Phi_x + ik\sigma_3\Phi = U(x,t)\Phi \\ \Phi_t + 2ik^2\sigma_3\Phi = V(x,t,k)\Phi \end{cases}$$

Direct problem: Determine two matrix solutions $\Phi_j(x,t,k)$, $j=1,2$ of the Lax pair equations imposing boundary conds. at $\pm\infty$ (Jost solutions):

$$\Phi_j(x,t,k) \sim N_j(k)e^{-(ikx+2ik^2t)\sigma_3}, \qquad x \to (-1)^{j+1}\infty,$$

where $N_1(k) = \begin{pmatrix} 1 & \frac{A}{2ik} \\ 0 & 1 \end{pmatrix}$ and $N_2(k) = \begin{pmatrix} 1 & 0 \\ \frac{A}{2ik} & 1 \end{pmatrix}$.

- $N_j(k)$ have singularities of the first order at $k=0$.

Being solutions of two ODEs, $\Phi_j(x,t,k)$, $j=1,2$ are related by scattering relation:

$$\Phi_1(x,t,k) = \Phi_2(x,t,k)S(k),\ k \in \mathbb{R}\setminus\{0\}.$$

Particularly, $S(k) = \Phi_2^{-1}(x,0,k)\Phi_1(x,0,k)$ and thus is uniquely determined by ini. conds. $q(x,0)$.

Properties of scattering matrix

$$\Phi_1(x,t,k) = \Phi_2(x,t,k)S(k),\ k \in \mathbb{R} \setminus \{0\}.$$

- Symmetry: from

$$\Lambda \overline{\Phi_1(-x,t,-\bar{k})}\Lambda^{-1} = \Phi_2(x,t,k), \qquad k \in \mathbb{R} \setminus \{0\},$$

where $\Lambda = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, it follows that $\Lambda \overline{S(-k)}\Lambda^{-1} = S^{-1}(k)$ and thus

$$S(k) = \begin{pmatrix} a_1(k) & -\overline{b(-k)} \\ b(k) & a_2(k) \end{pmatrix}, \qquad k \in \mathbb{R} \setminus \{0\},$$

where $a_1(k)$ and $a_2(k)$ are not related (important difference w.r.t. NLS).

1. $a_1(k)$ is analytic for $k \in \mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+} \setminus \{0\}$; $a_2(k)$ is analytic for $k \in \mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$.
2. $a_j(k) = 1 + O\left(\frac{1}{k}\right)$, $j = 1,2$; $\quad b(k) = O\left(\frac{1}{k}\right)$, $k \to \infty$.
3. $\overline{a_1(-\bar{k})} = a_1(k)$, $k \in \overline{\mathbb{C}^+} \setminus \{0\}$; $\quad \overline{a_2(-\bar{k})} = a_2(k)$, $k \in \overline{\mathbb{C}^-}$.
4. $a_1(k)a_2(k) + b(k)\overline{b(-\bar{k})} = 1$, $k \in \mathbb{R} \setminus \{0\}$.

$\Phi_j(x,t,k)$ can be determined via integral (Volterra) equations for $\Psi_j(x,t,k) := \Phi_j(x,t,k)e^{(ikx+2ik^2t)\sigma_3}$. From these integral equations one deduces the analytic (in k) and asymptotic (as $k\to\infty$ and as $k\to 0$) properties of Ψ_j:

as $k\to\infty$:

- $\Psi_1^{(1)}(x,t,k) = \begin{pmatrix}1\\0\end{pmatrix} + O(k^{-1}),\ \Psi_2^{(2)}(x,t,k) = \begin{pmatrix}0\\1\end{pmatrix} + O(k^{-1}),\ k\in\mathbb{C}^+,$
- $\Psi_1^{(2)}(x,t,k) = \begin{pmatrix}0\\1\end{pmatrix} + O(k^{-1}), \Psi_2^{(1)}(x,t,k) = \begin{pmatrix}1\\0\end{pmatrix} + O(k^{-1}),\ k\in\mathbb{C}^-,$

as $k\to 0$:

- $\Psi_1^{(1)} = \frac{1}{k}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix} + O(1),\ \Psi_1^{(2)} = \frac{2i}{A}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix} + O(k),$
- $\Psi_2^{(1)} = -\frac{2i}{A}\begin{pmatrix}\overline{v_2}(-x,t)\\\overline{v_1}(-x,t)\end{pmatrix} + O(k),\ \Psi_2^{(2)} = -\frac{1}{k}\begin{pmatrix}\overline{v_2}(-x,t)\\\overline{v_1}(-x,t)\end{pmatrix} + O(1).$

Generic and non-generic cases

As $k \to 0$, the spectral functions $a_1(k)$ and $b(k)$ behave as

$$a_1(k) = \frac{A^2 a_2(0)}{4k^2} + O\left(\frac{1}{k}\right), \qquad b(k) = \frac{A a_2(0)}{2ik} + O(1).$$

Thus we have two qualitatively different cases: (i) generic, with $a_2(0) \neq 0$ and (ii) non-generic, with $a_2(0) = 0$. The construction of the Riemann–Hilbert problem (the main tool for the inverse problem part of the IST method) is different in these cases.

"Pure step" initial data: $q(x,0) = 0$ for $x < 0$ and $q(x,0) = A$ for $x > 0$. In this case,

$$S(k) \equiv \begin{pmatrix} a_1(k) & -\overline{b(-k)} \\ b(k) & a_2(k) \end{pmatrix} = \begin{pmatrix} 1 + \frac{A^2}{4k^2} & -\frac{A}{2ik} \\ \frac{A}{2ik} & 1 \end{pmatrix}$$

Since $a_2(0) = 1$, "pure step" is in the generic case. Moreover, a "small perturbation" of the "pure step" initial data is also generic, with the following properties of zeros of spectral functions:

(i) $a_1(k)$ has a single, simple zero in $\overline{\mathbb{C}^+}$ (at some $k = ik_1$);
(ii) $a_2(k)$ has no zeros in $\overline{\mathbb{C}^-}$.

Define piecewise meromorphic matrix function $M(x,t,k)$ using the Jost solutions:

$$M(x,t,k)=\begin{cases}\left(\frac{\Psi_1^{(1)}(x,t,k)}{a_1(k)},\Psi_2^{(2)}(x,t,k)\right), & k\in\mathbb{C}^+\setminus\{0\},\\ \left(\Psi_2^{(1)}(x,t,k),\frac{\Psi_1^{(2)}(x,t,k)}{a_2(k)}\right), & k\in\mathbb{C}^-.\end{cases}$$

On the other hand, M **can be characterized** as solution of the Riemann–Hilbert problem with data uniquely determined by the ini. conds. $q(x,0)$ in terms of the associated spectral functions:

- Jump condition: $M_+(x,t,k)=M_-(x,t,k)J(x,t,k)$, $k\in\mathbb{R}\setminus\{0\}$, with the jump matrix

$$J(x,t,k)=\begin{pmatrix}1+r_1(k)r_2(k) & r_2(k)e^{-2ikx-4ik^2t}\\ r_1(k)e^{2ikx+4ik^2t} & 1\end{pmatrix},$$

  where $r_1(k)=\frac{b(k)}{a_1(k)}$ and $r_2(k)=\frac{\overline{b(-k)}}{a_2(k)}$.
- Normalization condition: $M(x,t,k)\to I$ as $k\to\infty$.

- Singularity conditions:
  - Residue condition at $k = ik_1$ (similar to local NLS):

$$\operatorname*{Res}_{k=ik_1} M^{(1)}(x,t,k) = \frac{\gamma_1}{\dot{a}_1(ik_1)} e^{-2k_1 x - 4ik_1^2 t} M^{(2)}(x,t,ik_1), |\gamma_1| = 1;$$

  - Conditions at $k = 0$ (specific for nonlocal NLS):

$$M_+ = \begin{pmatrix} \frac{4}{A^2 a_2(0)} v_1(x,t) & -\overline{v_2}(-x,t) \\ \frac{4}{A^2 a_2(0)} v_2(x,t) & -\overline{v_1}(-x,t) \end{pmatrix} (I + O(k)) \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix},$$

$$M_- = \frac{2i}{A} \begin{pmatrix} -\overline{v_2}(-x,t) & \frac{v_1(x,t)}{a_2(0)} \\ -\overline{v_1}(-x,t) & \frac{v_2(x,t)}{a_2(0)} \end{pmatrix} + O(k).$$

Let $M(x,t,k)$ be the solution of the RH problem. Then $q(x,t)$ can be expressed in terms of $M(x,t,k)$:

$$q(x,t) = 2i \lim_{k\to\infty} k M_{12}(x,t,k),$$

$$q(-x,t) = -2i \lim_{k\to\infty} k\overline{M_{21}(x,t,k)}.$$

Notice that it is sufficient to solve RHP for $x \geq 0$ only!

The basic tool of the large time analysis of the RH problem is the nonlinear steepest descent method (Deift and Zhou, 1993). The first step is the triangular factorizations of the jump matrix, in order to arrive at a "deformed" RHP with jump decaying (as $t \to \infty$) to I:

$$\begin{aligned} J(x,t,k) &= \begin{pmatrix} 1 & 0 \\ \frac{r_1 e^{2it\theta}}{1+r_1r_2} & 1 \end{pmatrix} \begin{pmatrix} 1+r_1r_2 & 0 \\ 0 & \frac{1}{1+r_1r_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{r_2 e^{-2it\theta}}{1+r_1r_2} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & r_2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1 e^{2it\theta} & 1 \end{pmatrix}, \end{aligned}$$

where

$$\theta(\xi,k) = 4k\xi + 2k^2, \ \xi = \frac{x}{4t}.$$

Due to the sign of $\operatorname{Re} i\theta$, use the lower/upper factorization for $k < -\xi$ and the upper/lower factorization for $k > -\xi$.

Large time analysis. Triangular factorizations, II

In order to get rid of the diagonal factor, determine $\delta(\xi, k)$ as the solution of the scalar RHP

$$\begin{cases} \delta_+(\xi,k) = \delta_-(\xi,k)(1 + r_1(k)r_2(k)), & k \in (-\infty; -\xi) \\ \delta(\xi,k) \to 1, & k \to \infty \end{cases}$$

- $1 + r_1(k)r_2(k)$ is complex-valued (real, for local NLS).
- We assume that $\int_{-\infty}^{-\xi} d\arg(1 + r_1(\zeta)r_2(\zeta)) \in (-\pi, \pi)$.

The solution of this scalar RHP is given by

$$\delta(\xi,k) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{-\xi} \frac{\ln(1 + r_1(\zeta)r_2(\zeta))}{\zeta - k}\, d\zeta\right\}$$

Determine $\tilde{M} := M\left(\begin{smallmatrix}\delta^{-1} & 0 \\ 0 & \delta\end{smallmatrix}\right)$. Then $\tilde{M}$ satisfies the RHP with jump

$$\tilde{J} = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)\delta_-^{-2}(\xi,k)}{1+r_1(k)r_2(k)}e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r_2(k)\delta_+^{2}(\xi,k)}{1+r_1(k)r_2(k)}e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k < -\xi \\ \begin{pmatrix} 1 & r_2(k)\delta^2(\xi,k)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1(k)\delta^{-2}(\xi,k)e^{2it\theta} & 1 \end{pmatrix}, & k > -\xi \end{cases}$$

Large time analysis. Deformed RHP

The next step is to multiply $\tilde{M}$ by triangular factors above and to deform the contour to the cross Γ centered at $k = -\xi$; in this way, $\tilde{M}(x,t,k) \rightsquigarrow \hat{M}(x,t,k)$ satisfying the RHP:

- $\hat{M}_+(x,t,k) = \hat{M}_-(x,t,k)\hat{J}(x,t,k),\ k \in \Gamma.$
- $\hat{M}(x,t,k) \to I,\ k \to \infty.$
- $\operatorname*{Res}_{k=ik_1} \hat{M}^{(1)}(x,t,k) = \frac{\gamma_1}{\dot{a}_1(ik_1)\delta^2(\xi,ik_1)} e^{-2k_1 x - 4ik_1^2 t} \hat{M}^{(2)}(x,t,ik_1).$
- $\operatorname*{Res}_{k=0} \hat{M}^{(2)}(x,t,k) = \frac{A\delta^2(\xi,0)}{2i} \hat{M}^{(1)}(x,t,0).$

Since $\hat{J}(x,t,k) \to I$ as $t \to \infty$ for all $k \neq -\xi$, and we consider $x > 0$ s.t. $x \to \infty$ when $t \to \infty$, a rough asymptotics follows (determined by the last res. cond.) for any $\xi \equiv x/4t > 0$ fixed:

$$q(x,t) = A\delta^2(\xi,0) + o(1), x > 0, \qquad q(x,t) = o(1), x < 0,$$

where $\delta^2(\xi,0) = \exp\left\{ \frac{1}{\pi i} \int_{-\infty}^{-\xi} \frac{\ln(1+r_1(\zeta)r_2(\zeta))}{\zeta}\, d\zeta \right\}$.

Large time analysis. Refined asymptotics

In order to refine the asymptotics: rescale the RHP "locally", near $k = -\xi$; this leads to a model RHP with a constant jump, that can be solved explicitly, in terms of the parabolic cylinder functions. The resulting asymptotics is as follows (for $\xi \equiv x/4t > 0$ fixed):

$$q(x,t) = t^{-\frac{1}{2} - \operatorname{Im}\nu(\xi)} \alpha_1(\xi) e^{4it\xi^2 - i\operatorname{Re}\nu(\xi)\ln t}(1 + o(1)), \quad x < 0,$$

$$q(x,t) = A\delta^2(\xi, 0) + t^{-\frac{1}{2} + \operatorname{Im}\nu(-\xi)} \alpha_2(\xi) e^{4it\xi^2 - i\operatorname{Re}\nu(-\xi)\ln t}(1 + o(1))$$
$$+ t^{-\frac{1}{2} - \operatorname{Im}\nu(-\xi)} \alpha_3(\xi) e^{-4it\xi^2 + i\operatorname{Re}\nu(-\xi)\ln t}(1 + o(1)), \quad x > 0,$$

where $\nu(\xi) = -\frac{1}{2\pi}\ln|1 + r_1(\xi)r_2(\xi)| - \frac{i}{2\pi}\int_{-\infty}^{\xi} d\arg(1 + r_1(\zeta)r_2(\zeta))$.

- Recall that we assume that $\operatorname{Im}\nu(\xi) \in (-\frac{1}{2}, \frac{1}{2})$.
- Uniform for $|\xi| \geq C$, for any $C > 0$.
- Connect the asymptotics for $\xi > 0$ and $\xi < 0$: open problem.

One-soliton solution

Let $b(k) \equiv 0$. Then

$$q_{sol}(x,t) = \frac{A}{1 - \gamma_1 e^{-Ax - iA^2 t}}$$

with any γ_1 s.t. $|\gamma_1| = 1$ is the solution (kink) of the NNLS

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0.$$

In this (non-generic) case, k_1 is uniquely determined as $k_1 = \frac{A}{2}$, and the corresponding spectral functions are

$$a_1(k) = \frac{k - i\frac{A}{2}}{k}, \quad a_2(k) = \frac{k}{k - i\frac{A}{2}}.$$

- The one-soliton solution is singular at $(x,t) = (0, t_n)$ with $t_n = \frac{\arg \gamma_1}{A^2} + \frac{2\pi n}{A^2}$, $n \in \mathbb{Z}$.

Thank you!